Geometrization and Generalization of the Kowalevski top

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Abstract

A new view on the Kowalevski top and the Kowalevski integration procedure is presented. For more than a century, the Kowalevski 1889 case, attracts full attention of a wide community as the highlight of the classical theory of integrable systems. Despite hundreds of papers on the subject, the Kowalevski integration is still understood as a magic recipe, an unbelievable sequence of skilful tricks, unexpected identities and smart changes of variables. The novelty of our present approach is based on our four observations. The first one is that the so-called fundamental Kowalevski equation is an instance of a pencil equation of the theory of conics which leads us to a new geometric interpretation of the Kowalevski variables w, x_1, x_2 as the pencil parameter and the Darboux coordinates, respectively. The second is observation of the key algebraic property of the pencil equation which is followed by introduction and study of a new class of **discriminantly separable polynomials**. All steps of the Kowalevski integration procedure are now derived as easy and transparent logical consequences of our theory of discriminantly separable polynomials. The third observation connects the Kowalevski integration and the pencil equation with the theory of multi-valued groups. The Kowalevski change of variables is now recognized as an example of a two-valued group operation and its action. The final observation is surprising equivalence of the associativity of the two-valued group operation and its action to $n = 3$ case of the Great Poncelet Theorem for pencils of conics.

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1 Introduction

The goal of this paper is to give a new view on the Kowalevski top and the Kowalevski integration procedure. For more than a century, the Kowalevski 1889 case [25], attracts the full attention of a wide community as the highlight of the classical theory of integrable systems. Despite hundreds of papers on the subject, the Kowalevski integration is still understood as a magic recipe, an unbelievable sequence of skilful tricks, unexpected identities and smart changes of variables (see for example [26], [24], [20], [29], [1], [11], [23], [17], [2], [4],[32], [18], [22], [27],[28], [14] and references therein).

The novelty of this paper is based on our four observations. The first one is that the so-called fundamental Kowalevski equation (see [25], [24], [20])

$$Q(w, x_1, x_2) = 0,$$

is an instance of a pencil equation from the theory of conics. This leads us to a new interpretation of the Kowalevski variables w, x_1, x_2 as the pencil parameter and the Darboux coordinates respectively. Origins and classical applications of the Darboux coordinates can be found in Darboux's book [9] , while some modern application can be found in [12], [13].

The second is observation of the key algebraic property of the pencil equation: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\mathcal{D}_w(Q)(x_1, x_2) = f_1(x_1)f_2(x_2)$$
$$\mathcal{D}_{x_1}(Q)(w, x_2) = f_3(w)f_2(x_2)$$
$$\mathcal{D}_{x_2}(Q)(w, x_1) = f_1(x_1)f_3(w)$$

This serves us as a motivation to introduce a new class of what we call **discriminantly separable polynomials**. We develop the theory of such polynomials. All steps of the Kowalevski integration now follow as easy and transparent logical consequences of our theory of the discriminantly separable polynomials.

The third observation connects the Kowalevski integration and the pencil equation with the theory of multivalued groups. The theory of multivalued groups started in the beginning of 1970's by Buchstaber and Novikov (see [5]). It has been further developed by Buchstaber and his collaborators in last forty years (see [8], [7], [6]). The Kowalevski change of variables is now recognized as a case of two-valued group operation $(\Gamma_2, \mathbb{Z}_2)$ and its action, where Γ_2 is an elliptic curve and $\mathbb{Z}_2$ its subgroup.

Our final observation is surprising equivalence of the associativity condition for this two-valued group operation to a case of the Great Poncelet Theorem for triangles. Well-known mechanical interpretation of the Great Poncelet Theorem is connected with integrable billiards, see for example [15]. The Great Poncelet Theorem is the milestone of the theory of pencils of conics and the whole classical projective geometry (see [30], and also [3], [15], [16] and references therein), as the Kowalevski top is the milestone of the classical integrable systems. Now

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we manage to relate them closely. As a consequence, we get a new connection between Great Poncelet Theorem and integrable mechanical systems, this time from rigid- body dynamics.

The paper is organized as follows. The next Section 2 starts with a subsection devoted to the pencils of conics and the Darboux coordinates. We derive the key property of the pencil equation-discriminant separability. In the second subsection, we formally introduce the class of discriminanlty separable polynomials and systematically study this class.

In the Section 3 we show how the Kowalevski case is embedded into our more general framework. A new geometric interpretation of the Kowalevski variables (w, x_1, x_2) as the pencil parameter and the Darboux coordinates is obtained.

In the Section 4 general systems are defined, related to the general equation of the pencil. The Kowalevski top can be seen as a special subcase. The first integrals are studied. Their properties are related to the properties of discriminantly separable polynomials, obtained in Section 2. It was done by use of what we call the Kotter trick (see [24], [20]). The nature of this transformation is going to be clarified in the last Section 5 through the theory of multivalued groups. Then, we manage to generalize another Kotter's transformation and this gives us a possibility to integrate the general system defined at the beginning of this Section. We reduce the problem to the functions $P_i, i = 1, 2, 3$. The evolution of those functions in terms of the theta-functions was obtained by Kowalevski herself in [25]. A modern account of the theta-functions and their applications to nonlinear equations one can find for example in [17].

The last Section 5 is devoted to two-valued groups and their connection with the Kowalevski top and the Great Poncelet Theorem. In order to make the text self-contained as much as possible, we start the Section with brief introduction to the theory of multivalued groups, following works of Buchstaber and his co-workers. The main role is played by two-valued coset group obtained from an elliptic curve Γ_2 and its subgroup $\mathbb{Z}_2$. It appears that the Kowalevski change of variables has its naturale expression through this two-valued group and its action. These results complete the picture obtained before by Weil in [33] and Jurdjevic [23]. Within this framework, we give an explanation of the Kotter trick, as we promised in Section 4. Finally, we show that the associativity condition for the two-valued group $(\Gamma_2, \mathbb{Z}_2)$ is equivalent to the famous Great Poncelet Theorem ([30]) in its basic $n = 3$ case.

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2 Pencils of conics and discriminantly separable polynomials

2.1 Pencils of conics and the Darboux coordinates

Let us start with two conics C_1 and C_2 given by their tangential equations:

$$C_1 : a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0;$$
$$C_2 : w_2^2 - 4w_1 w_3 = 0. \tag{1}$$

We assume that conics C_1 and C_2 are in general position. Consider the pencil $C(s)$ of conics $C_1 + sC_2$. The conics from the pencil share four common tangents. The coordinate equation of the conics of the pencil is:

$$F(s, z_1, z_2, z_3) := \det M(s, z_1, z_2, z_3) = 0, \tag{2}$$

where M is a bordered matrix of the form

$$M(s, z_1, z_2, z_3) = \begin{bmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & a_0 & a_1 & a_5 - 2s \\ z_2 & a_1 & a_2 + s & a_3 \\ z_3 & a_5 - 2s & a_3 & a_4 \end{bmatrix} \tag{3}$$

Then the point equation of the pencil of conics $C(s)$ is of the form of the quadratic polynomial in s

$$F := H + Ks + Ls^2 = 0 \tag{4}$$

where H, K and L are quadratic expressions in (z_1, z_2, z_3).

Following Darboux (see [9]), we introduce a new system of coordinates in the plane. Given a plane with standard coordinates (z_1, z_2, z_3), we start from the given conic C_2. The conic is given by the equation (1) and it is rationally parameterized by $(1, \ell, \ell^2)$. The tangent line to the conic C_2 through the point with the parameter ℓ_0 is given by the equation

$$t_{C_2}(\ell_0) : z_1 \ell_0^2 - 2z_2 \ell_0 + z_3 = 0.$$

On the other hand, for a given point P in the plane with coordinates $P = (\hat{z}_1, \hat{z}_2, \hat{z}_3)$ there correspond two solutions x_1 and x_2 of the equation quadratic in ℓ:

$$\hat{z}_1 \ell^2 - 2\hat{z}_2 \ell + \hat{z}_3 = 0. \tag{5}$$

Each solution corresponds to a tangent to the conic C_2 from the point P. We will call the pair (x_1, x_2) *the Darboux coordinates* of the point P. One finds immediately converse formulae

$$\hat{z}_1 = 1, \quad \hat{z}_2 = \frac{x_1 + x_2}{2}, \quad \hat{z}_3 = x_1 x_2. \tag{6}$$

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We change the variables in the polynomial F from projective coordinates $(z_1 : z_2 : z_3)$ to the Darboux coordinates according to the formulae 6. In the new coordinates we get the formulae:

$$
\begin{aligned}
H(x_1, x_2) =& (a_1^2 - a_0 a_2) x_1^2 x_2^2 + (a_0 a_3 - a_5 a_1) x_1 x_2 (x_1 + x_2) \\
& + (a_5^2 - a_0 a_4)(x_1^2 + x_2^2) + (2(a_5 a_2 - a_1 a_3) + \frac{1}{2}(a_5^2 - a_0 a_4) x_1 x_2 \\
& + (a_1 a_4 - a_3 a_5))(x_1 + x_2) + a_3^2 - a_2 a_4 \\
K(x_1, x_2) =& - a_0 x_1^2 x_2^2 + 2 a_1 x_1 x_2 (x_1 + x_2) - a_5 (x_1^2 + x_2^2) - 4 a_2 x_1 x_2 \\
& + 2 a_3 (x_1 + x_2) - a_4 \\
L(x_1, x_2) =& (x_1 - x_2)^2.
\end{aligned}
$$

$$(7)$$

We may notice for further references that

$$(x_1 - x_2)^2 = 4(z_1 z_3 - z_2^2). \tag{8}$$

Now, the polynomial

$$F(s, x_1, x_2) = L(x_1, x_2) s^2 + K(x_1, x_2) s + H(x_1, x_2)$$

is of the second degree in each of variables s, x_1 and x_2 and it is symmetric in (x_1, x_2). It has one **very exceptional property**, as described in the next theorem.

For a polynomial $P(y_1, y_2, \ldots, y_n)$ of variables $(y_1, y_2, \ldots, y_n)$ we will denote its discriminant with respect to the variable y_i by $\mathcal{D}_{y_i}(P)$ which is a polynomial of the rest of the variables $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$.

Theorem 1 *(i) There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial F in s as a polynomial in variables x_1 and x_2 separates the variables:*

$$\mathcal{D}_s(F)(x_1, x_2) = P(x_1) P(x_2). \tag{9}$$

(ii) There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial F in x_2 as a polynomial in variables x_1 and s separates the variables:

$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s) P(x_1). \tag{10}$$

Due to the symmetry between x_1 and x_2 the last statement remains valid after exchanging the places of x_1 and x_2.

Proof.

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(i) A general point belongs to two conics of a tangential pencil. If a point belongs to only one conic, then it belongs to one of the four common tangents of the pencil. At such a point, this unique conic touches one of the four common tangents. Thus, the equation

$$\mathcal{D}_s(F)(x_1, x_2) = 0 \tag{11}$$

which represents the condition of annulation of the discriminant, is the equation of the four common tangents. Thus, the equation 11 is equivalent to the system

$$x_1 = c_1 \quad x_1 = c_2 \quad x_1 = c_3 \quad x_1 = c_4$$
$$x_2 = c_1 \quad x_2 = c_2 \quad x_2 = c_3 \quad x_2 = c_4$$

where c_i are parameters which correspond to the points of contact of the four common tangents with the conic C_2. As a consequence, we get

$$\mathcal{D}_s(F)(x_1, x_2) = P(x_1)P(x_2),$$

where the polynomial P is of the fourth degree and of the form

$$P(x) = a(x - c_1)(x - c_2)(x - c_3)(x - c_4).$$

This proves the first part of the theorem.

The second part of the Theorem follows from the following Lemma.

Lemma 1 *Given a polynomial $S = S(x, y, z)$ of the second degree in each of its variables in the form:*

$$S(x, y, z) = A(y, z)x^2 + 2B(y, z)x + C(y, z).$$

If there are polynomials P_1 and P_2 of the fourth degree such that

$$B(y, z)^2 - A(y, z)C(y, z) = P_1(y)P_2(z), \tag{12}$$

then there exists a polynomial f such that

$$\mathcal{D}_y S(x, z) = f(x)P_2(z), \quad D_z S(x, y) = f(x)P_1(y).$$

Proof. To prove the Lemma, rewrite the equation (12) in the equivalent form

$$(B + uA)^2 - A(u^2 A + 2uB + C) = P_1(y)P_2(z).$$

For a zero $y = y_0$ of the polynomial P_1, any zero of $S(u, y_0, z)$ as a polynomial in z is a double zero, according to the last equation. Thus, y_0 is a zero of $D_z S(x, y)$. Thus, the polynomial P_1 is a factor of the polynomial $D_z S(x, y)$. Since the degree of the polynomial P_1 is four, then there exists a polynomial f in x such that

$$D_z S(x, y) = f(x)P_1(y).$$

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The rest of the Lemma follows by double application of the same arguments. $\qquad\square$

(ii) Now, the proof of the second part of the Theorem 1 follows by immediate application of the Lemma 1

$\qquad\square$

Proposition 1 *(i) The explicit formulae for the polynomials P and J are*

$$P(x) = a_0 x^4 - 4a_1 x^3 + (2a_5 + 4a_2)x^2 - 4a_3 x + a_4$$
$$J(s) = -4s^3 + 4(a_5 - a_2)s^2 + (a_0 a_4 - a_5^2 + 4(a_5 a_2 - a_1 a_3))s \qquad (13)$$
$$- a_3^2 a_0 + a_0 a_4 a_2 + 2a_1 a_3 a_5 - a_4 a_1^2 - a_2 a_5^2.$$

(ii) If all the zeros of the polynomial P are simple, then the elliptic curves

$$\Gamma_1 : y^2 = P(x)$$
$$\Gamma_2 : t^2 = J(s)$$

are isomorphic and the later can be understood as Jacobian of the former.

Proof. Instead of straightforward calculation, we are going to consider a double-bordered determinant (see [9], [31], [21]) obtained from the matrix M (3):

$$\hat{M} = \begin{vmatrix} 0 & 0 & z_1' & z_2' & z_3' \\ 0 & 0 & z_1 & z_2 & z_3 \\ z_1' & z_1 & a_0 & a_1 & a_5 - 2s \\ z_2' & z_2 & a_1 & a_2 + s & a_3 \\ z_3' & z_3 & a_5 - 2s & a_3 & a_4 \end{vmatrix} \qquad (14)$$

We apply the Jacobi identity and get

$$\hat{M}_{11}\hat{M}_{22} - (\hat{M}_{12})^2 = \hat{M}\hat{M}_{12,12}.$$

Obviously, $\hat{M}_{12,12}$ is a polynomial only in s of the third degree:

$$\hat{M}_{12,12} = -4s^3 + 4(a_5 - a_2)s^2 + ((a_0 a_4 - a_5^2) + 4(a_5 a_2 - a_1 a_3))s$$
$$+ a_0 a_4 a_2 - a_3^2 a_0 + 2a_1 a_3 a_5 - a_4 a_1^2 - a_2 a_5^2$$
$$= J(s)$$

Moreover, if we substitute

$$z_1 = 1, \quad z_2 = \frac{x_1 + x_2}{2}, \quad z_3 = x_1 x_2$$
$$z_1' = 1, \quad z_2' = \frac{x_1 + x_2'}{2}, \quad z_3' = x_1 x_2'$$

we have

$$\hat{M} = P(x_1)\frac{(x_2 - x_2')^2}{4}$$
$$\hat{M}_{11} = F(s, x_1, x_2)$$
$$\hat{M}_{22} = F(s, x_1, x_2').$$

If we denote

$$F(s, x_1, x_2) = T(s, x_1)x_2^2 + V(s, x_1)x_2 + W(s, x_1)$$

then

$$\hat{M}_{12} = Tx_2x_2' + V\frac{x_2 + x_2'}{2} + W.$$

From the last equations, after dividing by $(x_2 - x_2')^2$, we get

$$V^2 - 4TW = J(s)P(x_1),$$

and the proof of the first part of the Proposition is finished.

The second part follows by direct calculation of correspondence between two elliptic curves, one of which is defined by a polynomial of degree 3 and one by polynomial of degree 4. $\qquad\square$

2.2 Discriminantly separable polynomials

We saw that a polynomial of three variables which defines a pencil of conics has a very peculiar property: all three of its discriminants are representable as products of two polynomials of one variable each. These considerations motivate the following definition.

Definition 1 *For a polynomial $F(x_1, \ldots, x_n)$ we say that it is discriminantly separable if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \ldots, n$*

$$\mathcal{D}_{x_i}F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is symmetrically discriminantly separable if

$$f_2 = f_3 = \cdots = f_n,$$

while it is strongly discriminatly separable if

$$f_1 = f_2 = f_3 = \cdots = f_n.$$

It is weakly discriminantly separable if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \ldots, n$

$$\mathcal{D}_{x_i}F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

Theorem 2 *Given a polynomial $F(s, x_1, x_2)$ of the second degree in each of the variables s, x_1, x_2 of the form*

$$F = s^2 A(x_1, x_2) + 2B(x_1, x_2)s + C(x_1, x_2).$$

Denote by T_{B^2-AC} a 5×5 matrix such that

$$(B^2 - AC)(x_1, x_2) = \sum_{j=1}^{5} \sum_{i=1}^{5} T_{B^2-AC}^{ij} x_1^{i-1} x_2^{j-1}.$$

Then, polynomial F is discriminantly separable if and only if

$$\operatorname{rank} T_{B^2-AC} = 1.$$

Proof. The proof follows from the Lemma 1 and the observation that a polynomial in two variables is equal to a product of two polynomials in one variable if and only if its matrix is equal to a tensor product of two vectors. The last condition is equivalent to the condition on rank of the last matrix to be equal to 1. $\qquad\square$

Proposition 2 *Given a polynomial $F(s, x_1, x_2)$ of the second degree in each of the variables s, x_1, x_2. of the form*

$$F = s^2 A(x_1) + 2B(x_1, x_2)s + C(x_2),$$

where A depends only on x_1 and C depends only on x_2. Denote by T_{B^2} a 5×5 matrix such that

$$(B^2)(x_1, x_2) = \sum_{i=1}^{5} T_{B^2}^{ij} x_1^{i-1} x_2^{j-1}.$$

Then, polynomial F is discriminantly separable if and only if

$$\operatorname{rank} T_{B^2} = 2.$$

Proof. The proof follows from the observation of the proof of the last theorem and a fact that a matrix of rank two is equal to a sum of two matrices of rank one. $\qquad\square$

The last Proposition gives a method to construct nonsymmetric discriminantly separable polynomials.

Lemma 2 *Given an arbitrary quadratic polynomial*

$$F = s^2 A + 2Bs + C.$$

Then, the square of its differential is equal to its discriminant under the condition $F = 0$:

$$\left(\frac{dF}{ds}\right)^2 = 4(B^2 - AC).$$

Corollary 1 *For an arbitrary discriminantly separable polynomial $F(x_3, x_1, x_2)$ of the second degree in each of the variables x_3, x_1, x_2, its differential is separable on the surface $F(x_3, x_1, x_2) = 0$:*

$$\frac{dF}{\sqrt{f_3(x_3)f_1(x_1)f_2(x_2)}} = \frac{dx_3}{\sqrt{f_3(x_3)}} + \frac{dx_1}{\sqrt{f_1(x_1)}} + \frac{dx_2}{\sqrt{f_2(x_2)}}.$$

The proof of the corollary is straightforward application of the previous statements. This property of discriminantly separable polynomials is fundamental in their role in the theory of integrable systems. Observe that analogous statement is valid for arbitrary discriminantly separable polynomials.

From the last Corollary, applied to a symmetric discriminatly separable polynomial of the second degree, immediately follows a variant of the Euler theorem.

Corollary 2 *The condition $x_3 = \text{const}$ defines a conic from the pencil as an integral curve of the Euler equation:*

$$\frac{dx_1}{\sqrt{f_1(x_1)}} + \frac{dx_2}{\sqrt{f_1(x_2)}} = 0,$$

where f_1 is general polynomial of degree 4.

Proposition 3 *All symmetric discriminantly separable polynomials $F(s, x_1, x_2)$ of degree two in each variable with the leading coefficient*

$$L(x_1, x_2) = (x_1 - x_2)^2$$

are of the form

$$F(s, x_1, x_2) = (x_1 - x_2)^2 s^2 + K(x_1, x_2)s + H(x_1, x_2)$$

where K and H are done by the formulae (7).

The next Lemma gives a possibility to create new discriminantly separable polynomials from a given one.

Lemma 3 *Given a discriminantly separable polynomial*

$$F(s, x_1, x_2) := A(x_1, x_2)s^2 + 2B(x_1, x_2)s + C(x_1, x_2)$$

of the second degree in each variable.

(a) Let $\alpha(x)$ be a linear transformation. Then polynomial

$$F_1(s, x_1, x_2) := F(s, \alpha(x_1), x_2)$$

is discriminantly separable.

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(b) The polynomial

$$\hat{F}(s, x_1, x_2) := C(x_1, x_2)s^2 + 2B(x_1, x_2)s + A(x_1, x_2)$$

is discriminantly separable.

The transformation from F to $\hat{F}$ described in the Lemma 3 (b) maps a solution s of the equation $F = 0$ to $1/s$. We will use the term *transposition* for such a transformation from F to $\hat{F}$. Thus, summarizing we get

Corollary 3 *Given a discriminantly separable polynomial*

$$F(s, x_1, x_2) := A(x_1, x_2)s^2 + 2B(x_1, x_2)s + C(x_1, x_2)$$

of the second degree in each variable and three fractionally-linear transformations α, β, γ. Then the polynomial

$$F_1(s, x_1, x_2) := F(\gamma(s), \alpha(x_1), \beta(x_2))$$

is discriminantly separable.

From the last Lemma we have a procedure to create non-symmetric discriminantly separable polynomials from a given symmetric discriminantly separable polynomial. The converse statement is also true:

Proposition 4 *Given a discriminantly separable polynomial*

$$F(s, x_1, x_2) := A(x_1, x_2)s^2 + 2B(x_1, x_2)s + C(x_1, x_2)$$

of the second degree in each variable. Suppose that a biquadratic $F(s_0, x_1, x_2)$ is nondegenerate for some value $s = s_0$. Then there exists a fractionally-linear transformations α such that the polynomial

$$F_1(s, x_1, x_2) := F(s, \alpha(x_1), x_2)$$

is symmetrically discriminantly separable.

Proof. Let us fix an arbitrary value for s such that $B(x_1, x_2)$ is a nondegenerate biquadratic. Keeping s fixed, we have a relation

$$\frac{dx_1}{\sqrt{f_1(x_1)}} \pm \frac{dx_2}{\sqrt{f_2(x_2)}} = 0,$$

where f_1, f_2 are two polynomials, each in one variable. For a given x_1 there are two corresponding points x_2 and $\hat{x}_2$. The last two are connected by the relation

$$\frac{d\hat{x}_2}{\sqrt{f_2(\hat{x}_2)}} \pm \frac{dx_2}{\sqrt{f_2(x_2)}} = 0,$$

where now denominator of both fraction is one and the same polynomial, f_2. This means that there exists an elliptic function u of degree two and a shift T on the elliptic curve $y^2 = f_2(x)$, such that x_2 and $\hat{x}_2$ are parameterized by

$$x_2 = u(z) \quad \hat{x}_2 = u(z + T).$$

From the relations
$$B(x_1, x_2) = 0 \quad B(x_1, \hat{x}_2) = 0$$

we see that both y and y^2 are elliptic functions of degree at most four which can be expressed through $x_2, \hat{x}_2$. Thus, y is an elliptic function of degree two. There is a fractional-linear transformation which reduces y to $u(z + T/2)$. This concludes the proof of the Proposition. $\qquad\square$

3 Geometric interpretation of the Kowalevski fundamental equation

The magic integration of the Kowalevski top is based on the Kowalevski fundamental equation, see [24], [20]:

$$Q(w, x_1, x_2) := (x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2) = 0, \qquad (15)$$

where

$$\begin{aligned}
R(x_1, x_2) &= -x_1^2 x_2^2 + 6l_1 x_1 x_2 + 2lc(x_1 + x_2) + c^2 - k^2 \\
R_1(x_1, x_2) &= -6l_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4clx_1 x_2(x_1 + x_2) \\
&\quad + 6l_1(c^2 - k^2) - 4c^2 l^2.
\end{aligned} \qquad (16)$$

If we replace in the equations (4) and (7) the following values for the coefficients:

$$\begin{aligned}
a_0 &= -2 \quad a_1 = 0 \quad a_5 = 0 \\
a_2 &= 3l_1 \quad a_3 = -2cl \quad a_4 = 2(c^2 - k^2)
\end{aligned} \qquad (17)$$

and compare with (15) and (16), we get the following

Theorem 3 *The Kowalevski fundamental equation represents a point pencil of conics given by their tangential equations*

$$\begin{aligned}
\hat{C}_1 &: -2w_1^2 + 3l_1 w_2^2 + 2(c^2 - k^2)w_3^2 - 4clw_2 w_3 = 0; \\
C_2 &: w_2^2 - 4w_1 w_3 = 0.
\end{aligned} \qquad (18)$$

The Kowalevski variables w, x_1, x_2 in this geometric settings are the pencil parameter, and the Darboux coordinates with respect to the conic C_2 respectively.

The Kowalevski case corresponds to the general case under the restrictions

$$a_1 = 0 \quad a_5 = 0 \quad a_0 = -2.$$

The last of these three relations is just normalization condition, provided $a_0 \neq 0$. The Kowalevski parameters l_1, l, c are calculated by the formulae

$$l_1 = \frac{a_2}{3} \quad l = \pm\frac{1}{2}\sqrt{-a_4 + \sqrt{a_4 + 4a_3^2}} \quad c = \mp\frac{a_3}{\sqrt{-a_4 + \sqrt{a_4 + 4a_3^2}}}$$

provided that l and c are requested to be real. Let us mention at the end of this Section, that in the original paper [25], instead the relation (15), Kowalevski used the equivalent one

$$\hat{Q}(s, x_1, x_2) := (x_1 - x_2)^2(s - \frac{l_1}{2})^2 - R(x_1, x_2)(s - \frac{l_1}{2}) - \frac{R_1(x_1, x_2)}{4} = 0.$$

The equivalence is obtained by putting $w = 2s - l_1$.

4 Generalized integrable system

4.1 Equations of motion and the first integrals

We are going to consider the following system of differential equations on unknown functions e_1, e_2, x_1, x_2, r, g:

$$\frac{de_1}{dt} = -\alpha e_1$$

$$\frac{de_2}{dt} = \alpha e_2$$

$$\frac{dx_1}{dt} = -\beta(rx_1 + cg)$$

$$\frac{dx_2}{dt} = \beta(rx_2 + cg)$$

$$\frac{dr}{dt} = -\beta(x_2 - x_1)(x_1 + x_2 + a_1) - \frac{\alpha}{2r}(e_1 - e_2)$$

$$\frac{dg}{dt} = \frac{\beta}{2c}\left[(x_2 - x_1)(x_1 x_2 - a_5) + e_1 x_2 - e_2 x_1\right] + \frac{(2r\beta - \alpha)}{2c^2 g}\left(e_1 x_2^2 - e_2 x_1^2\right)$$

(19)

Here β and α are given functions of e_1, e_2, x_1, x_2, r, g. The choice of their form defines different systems. The Kowalevski top is equivalent to the above system for

$$a_1 = 0 \quad a_5 = 0,$$

with the choice

$$\alpha = ir \quad \beta = \frac{i}{2}.$$

(20)

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We will assume in what follows that a_1 and a_5 are general. Beside the last choice for α and β, there are many others choices which also provide polynomial vector fields, such as (A) $\alpha = kr^2 \quad \beta = \frac{k}{2}r$, (B) $\alpha = krg \quad \beta = k_1g$, (C) $\alpha = kr^2g \quad \beta = k_1g$. Interesting cases satisfy the system (38) from Proposition (8).

Proposition 5 *The system (19) has the following first integrals*

$$
\begin{aligned}
k^2 &= e_1 \cdot e_2 \\
a_0a_2 &= e_1 + e_2 - (x_1 + x_2)^2 - 2a_1(x_1 + x_2) - r^2 \\
-\frac{a_0a_3}{2} &= -x_2e_1 - x_1e_2 + x_1x_2(x_1 + x_2) + \frac{a_5}{2}(x_1 + x_2) + a_1x_1x_2 - rg \\
\frac{a_0a_4}{4} &= x_2^2e_1 + x_1^2e_2 - x_1^2x_2^2 - a_5x_1x_2 - g^2
\end{aligned}
\tag{21}
$$

One can rewrite the last relations in the following form

$$
\begin{aligned}
k^2 &= e_1 \cdot e_2 \\
r^2 &= e_1 + e_2 + \hat{E}(x_1, x_2) \\
rg &= -x_2e_1 - x_1e_2 + \hat{F}(x_1, x_2) \\
g^2 &= x_2^2e_1 + x_1^2e_2 + \hat{G}(x_1, x_2),
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\hat{E}(x_1, x_2) &= -a_0a_2 - K(x_1 + x_2)^2 - 2a_1(x_1 + x_2) \\
\hat{F}(x_1, x_2) &= \frac{a_0a_3}{2} + Kx_1x_2(x_1 + x_2) + \frac{a_5}{2}(x_1 + x_2) + a_1x_1x_2 \\
\hat{G}(x_1, x_2) &= -\frac{a_0a_4}{4} - Kx_1^2x_2^2 - a_5x_1x_2,
\end{aligned}
\tag{23}
$$

with

$$
K = 1.
$$

Lemma 4 *If the polynomials $\hat{E}, \hat{F}, \hat{G}$ are defined by the equation (23) then the polynomial*

$$
P(x_1) := \hat{E}(x_1, x_2)x_1^2 + 2\hat{F}(x_1, x_2)x_1 + \hat{G}(x_1, x_2)
$$

depends only on x_1.

Proposition 6 *Given three polynomials $\hat{E}(x_1, x_2), \hat{F}(x_1, x_2), \hat{G}(x_1, x_2)$ of the second degree in each variable such that*

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(1) *Polynomials P, Q defined by*

$$P(x_1) := \hat{E}(x_1, x_2)x_1^2 + 2\hat{F}(x_1, x_2)x_1 + \hat{G}(x_1, x_2)$$
$$Q(x_2) := \hat{E}(x_1, x_2)x_2^2 + 2\hat{F}(x_1, x_2)x_2 + \hat{G}(x_1, x_2)$$
(24)

depend only on one variable each.

(2) *Polynomials $R(x_1, x_2)$ and $R_1(x_1, x_2)$ defined by*

$$R(x_1, x_2) := \hat{E}(x_1, x_2)x_1x_2 + \hat{F}(x_1, x_2)(x_1 + x_2) + \hat{G}(x_1, x_2)$$
$$R_1(x_1, x_2) := \hat{E}(x_1, x_2)\hat{G}(x_1, x_2) - \hat{F}^2(x_1, x_2)$$
(25)

are of the second degree in each variables.

Then:

(a) *The polynomials $\hat{E}(x_1, x_2), \hat{F}(x_1, x_2), \hat{G}(x_1, x_2)$ are symmetric in x_1, x_2.*

(b) *The polynomial*

$$F(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2)$$

is discriminantly separable.

(c) *The most general form of the polynomials $\hat{E}, \hat{F}, \hat{G}$ is given in the equation (23), with K arbitrary.*

(d) *For $K = 1$ the polynomial P is the one given in the Proposition 1.*

Proof. The proof follows by straightforward calculation with application of the Lemma 1. $\qquad\square$

If the coefficient K is nonzero we may normalize it to be equal to one. Under this assumption, the equations (23) with $K = 1$ are general. The case $K = 0$ is going to be analyzed separately in one of the following sections.

From the equations (22) we get the following

Corollary 4 *The relation is satisfied*

$$e_2 P(x_1) + e_1 P(x_2) - H(x_1, x_2) + k^2(x_1 - x_2)^2 = 0.$$
(26)

where P is the polynomial defined in the Lemma 4.

Corollary 5 *The differentials of x_1 and x_2 may be written in the form*

$$\frac{dx_1}{dt} = -\beta\sqrt{P(x_1) + e_1(x_1 - x_2)^2}$$
$$\frac{dx_2}{dt} = \beta\sqrt{P(x_2) + e_2(x_1 - x_2)^2}.$$
(27)

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The proof follows from the equations (22) and Lemma 4. Now, we apply what we are going to call *the Kotter trick*:

$$\left[\sqrt{e_1}\frac{\sqrt{P(x_2)}}{x_1-x_2} \pm \sqrt{e_2}\frac{\sqrt{P(x_1)}}{x_1-x_2}\right]^2 = (w_1 \pm k)(w_2 \mp k), \qquad (28)$$

where w_1, w_2 are solutions of the quadratic equation

$$F(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2). \qquad (29)$$

The Kotter trick appeared in [24] quite mysteriously. Further explanation done by Golubev sixty years later seems to be even trickier, see [20] and much less clear. In the last section of this paper, see Proposition 11, we provide a new interpretation of this transformation as a commuting diagram of morphisms of double-valued group. Should we hope that our explanation is more transparent then previous ones, since new sixty years passed in meantime?

From the last relations, following Kotter, one gets

$$\left(\frac{dx_1}{\sqrt{P(x_1)}dt}\right)^2 = \beta^2\left(1 + \frac{(x_1-x_2)^4 e_1 P(x_2)}{P(x_1)P(x_2)(x_1-x_2)^2}\right)$$

$$= \beta^2\left(1 + \frac{(\sqrt{(w_1-k)(w_2+k)} + \sqrt{(w_1+k)(w_2-k)})^2}{(w_1-w_2)^2}\right)$$

$$\left(\frac{dx_2}{\sqrt{P(x_2)}dt}\right)^2 = \beta^2\left(1 + \frac{(x_1-x_2)^4 e_2 P(x_1)}{P(x_1)P(x_2)(x_1-x_2)^2}\right)$$

$$= \beta^2\left(1 + \frac{(\sqrt{(w_1-k)(w_2+k)} - \sqrt{(w_1+k)(w_2-k)})^2}{(w_1-w_2)^2}\right).$$

Next, we get

$$\frac{dx_1}{\sqrt{P(x_1)}dt} = -\beta\left(\frac{\sqrt{(w_1-k)(w_1+k)} + \sqrt{(w_2+k)(w_2-k)}}{(w_1-w_2)}\right)$$
$$\frac{dx_2}{\sqrt{P(x_2)}dt} = -\beta\left(\frac{\sqrt{(w_1-k)(w_1+k)} - \sqrt{(w_2+k)(w_2-k)}}{(w_1-w_2)}\right) \qquad (30)$$

Now we apply the discriminant separability property of the polynomial F:

$$\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} = \frac{dw_1}{\sqrt{J(w_1)}}$$
$$\frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} = \frac{dw_2}{\sqrt{J(w_2)}} \qquad (31)$$

We will refer to the last relations as *the Kowalevski change of variables*. The nature of these relations has been studied by Jurdjevic (see [23]) following Weil ([33]). We are going to develop further these efforts in the Section 5 where we are going to show that the Kowalevski change of variables is the infinitesimal version of a double valued group operation and its action. From the relations 31 and 30 we finally get:

$$\frac{dw_1}{\sqrt{\Phi(w_1)}} + \frac{dw_2}{\sqrt{\Phi(w_2)}} = 0$$

$$\frac{w_1\,dw_1}{\sqrt{\Phi(w_1)}} + \frac{w_2\,dw_2}{\sqrt{\Phi(w_2)}} = 2\beta\,dt, \tag{32}$$

where

$$\Phi(w) = J(w)(w-k)(w+k),$$

is the polynomial of fifth degree. Thus, the equations (32) represent the Abel-Jacobi map of the genus 2 curve

$$y^2 = \Phi(w).$$

4.2 Generalized Kotter transformation

In order to integrate the dynamics on the Jacobian of the hyper-elliptic curve $y^2 = \Phi(w)$ we are going to generalize classical Kotter transformation. In this section we will assume the normalization condition

$$a_0 = -2.$$

Proposition 7 *For the polynomial $F(s, x_1, x_2)$ there exist polynomials $A_0(s)$, $f(s)$, $A(s, x_1, x_2)$, $B(s, x_1, x_2)$ such that the following identity*

$$F(s, x_1, x_2) \cdot A_0(s) = A^2(s, x_1, x_2) + f(s) \cdot B(s, x_1, x_2), \tag{33}$$

is satisfied. The polynomials are defined by the formulae:

$$A(s, x_1, x_2) = A_0(s)(x_1 x_2 - s) + B_0(s)(x_1 + x_2) + M_0(s)$$

$$A_0(s) = a_1^2 - a_0 a_2 - s a_0$$

$$B_0(s) = \frac{1}{2}(a_0 a_3 - a_5 a_1 + 2 s a_1)$$

$$M_0(s) = a_5 a_2 - a_1 a_3 + s(a_1^2 + a_5)$$

$$B(s, x_1, x_2) = (x_1 + x_2)^2 + 2a_1(x_1 + x_2) - 2s - 2a_2$$

$$f(s) = 2s^3 + 2(a_2 - a_5)s^2 + \left(2(a_1 a_3 - a_5 a_2) + a_4 + \frac{a_5^2}{2}\right)s + f_0$$

$$f_0 = a_4 a_2 - a_3^2 - a_1 a_3 a_5 + \frac{a_4 a_1^2 + a_2 a_5^2}{2}.$$

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For $a_5 = a_1 = 0$ the previous identity has been obtained in [24]. Following Kotter's idea, consider the identity

$$F(s) = F(u) + (s - u)F'(u) + (s - u)^2.$$

From the last two identities we get a quadratic equation in $s - u$

$$(s - u)^2(x_1 - x_2)^2 - 2(s - u)(R(x_1, x_2) - u(x_1 - x_2)) + f(u)B + (x_1 - x_2)^2 A^2.$$

Corollary 6 *(a) The solutions of the last equation satisfy the identity in u:*

$$(s_1 - u)(s_2 - u) = \frac{A^2}{(x_1 - x_2)^2} + f(u)\frac{B}{(x_1 - x_2)^2}.$$

(b) Denote m_1, m_2, m_3 the zeros of the polynomial f, and

$$P_i = \sqrt{(s_1 - m_i)(s_2 - m_i)}, \quad i = 1, 2, 3.$$

Then

$$P_i = \frac{1}{x_1 - x_2}\left(\sqrt{A_0(m_i)}x_1 x_2 + \frac{B_0(m_i)}{\sqrt{A_0(m_i)}} + m_i(m_i - a_5 - 2a_2) - 2a_5 - a_1 a_3\right), \quad i = 1, 2, 3.$$

$$(34)$$

Now we introduce more convenient notation

$$n_i = m_i + a_1^2 + 2a_2, \quad i = 1, 2, 3;$$

$$X = \frac{x_1 x_2 + (2a_1^2 + a_5 + 2a_2) + \frac{a_1}{2}(x_1 - x_2)}{x_1 - x_2},$$

$$Y = \frac{1}{x_1 - x_2},$$

$$Z = \frac{(a_1^3 + 2a_2 a_1 + 2a_5 a_1 + 2a_3)(x_1 + x_2) - 2(a_1^2 + 2a_2)(a_1^2 + a_5)}{x_1 - x_2}.$$

Lemma 5 *The quantities X, Y, Z satisfy the system of linear equations*

$$X - n_1 Y + \frac{1}{2n_1}Z = \frac{P_1}{\sqrt{n_1}}$$

$$X - n_2 Y + \frac{1}{2n_2}Z = \frac{P_2}{\sqrt{n_2}} \qquad (35)$$

$$X - n_3 Y + \frac{1}{2n_3}Z = \frac{P_3}{\sqrt{n_3}}.$$

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Denote
$$\hat{f}(x) = f(x - a_1^2 - 2a_2).$$
One can easily solve the previous linear system and get

Lemma 6 *The solutions of the system (35) are*

$$Y = -\left(\frac{P_1\sqrt{n_1}}{\hat{f}'(n_1)} + \frac{P_2\sqrt{n_2}}{\hat{f}'(n_2)} + \frac{P_3\sqrt{n_3}}{\hat{f}'(n_3)} \right)$$

$$Z = 2n_1 n_2 n_3 \left(\frac{P_1}{\sqrt{n_1}\hat{f}'(n_1)} + \frac{P_2}{\sqrt{n_2}\hat{f}'(n_2)} + \frac{P_3}{\sqrt{n_3}\hat{f}'(n_3)} \right)$$

The expression in terms of theta functions for $P_i = \sqrt{(s_1 - m_i)(s_2 - m_i)}$ for $i = 1, 2, 3$ can be obtained from [25] paragraph 7.

4.3 Interpretation of the equations of motion

Rigid-body coordinates

We are going to present briefly the interpretation of the equations of motion (19) in the standard rigid-body coordinates $p, q, r, \gamma, \gamma', \gamma''$, where:

$$e_1 = x_1^2 + c(\gamma + i\gamma')$$
$$e_2 = x_2^2 + c(\gamma - i\gamma')$$
$$p = \frac{x_1 + x_2}{2}$$
$$q = \frac{x_1 - x_2}{2i}.$$

From the last four equations of the system (19) we get

$$\dot{p} = - i\beta rq$$
$$\dot{q} = i\beta rp$$
$$\dot{r} = 2\beta iq(2p + a_1) - \frac{i\alpha}{r}(2pq + c\gamma')$$
$$\dot{\gamma}'' = - \frac{\beta}{c}(qia_5 + 2ic\gamma q - 2ic\gamma' p)$$
$$\quad + \frac{2r\beta - \alpha}{c^2\gamma''}(ic\gamma'(p^2 - q^2) - 2icpq\gamma)$$

(36)

while the equations for $\dot{\gamma}, \dot{\gamma}'$ can easily be obtained from the first two equations of the system (19):

$$\dot{\gamma} = \frac{\alpha}{2c}(x_2^2 - x_1^2) - i\alpha\gamma' + \frac{-x_1\dot{x}_1 - x_2\dot{x}_2}{c}$$
$$\dot{\gamma}' = \frac{\alpha}{2c}(-x_2^2 - x_1^2) - i\alpha\gamma + \frac{-x_1\dot{x}_1 + x_2\dot{x}_2}{c}.$$

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Finally, we get

$$\dot{\gamma} = \frac{2i(2\beta r - \alpha)}{c}pq - i\alpha\gamma' + 2i\beta\gamma''q$$

$$\dot{\gamma}' = -\frac{2i(2\beta r - \alpha)}{c}(p^2 - q^2) + i\alpha\gamma - 2i\beta\gamma''q \qquad (37)$$

Proposition 8 *The system (36, 37) preserves the standard measure if and only if*

$$A_0\alpha + A_1\alpha_p + A_2\alpha_q + A_3\alpha_r + A_4\alpha_\gamma + A_5\alpha_{\gamma'} + A_6\alpha_{\gamma''} +$$
$$B_0\beta + B_1\beta_p + B_2\beta_q + B_3\beta_r + B_4\beta_\gamma + B_5\beta_{\gamma'} + B_6\beta_{\gamma''} = 0, \qquad (38)$$

where

$A_0 = r^2\gamma'p^2 + c^2\gamma''^2\gamma' - 2r^2pq\gamma + 2c\gamma''^2pq - r^2\gamma'q^2$

$A_1 = 0$

$A_2 = 0$

$A_3 = -2c\gamma''^2rpq - c^2\gamma''^2r\gamma'$

$A_4 = -2pqr^2\gamma''^2 - \gamma'r^2c\gamma''^2$

$A_5 = -2r^2\gamma''^2q^2 + gr^2c\gamma''^2 + 2r^2\gamma''^2p^2$

$A_6 = -r^2\gamma''\gamma'p^2 + 2r^2\gamma''pq\gamma + r^2\gamma''\gamma'q^2$

$B_0 = -2r^3\gamma'p^2 + 2r^3\gamma'q^2 + 4r^3pq\gamma$

$B_1 = -cr^3q\gamma''^2$

$B_2 = cr^3p\gamma''^2$

$B_3 = 4qr^2c\gamma''^2p + 2qr^2c\gamma''^2a_1$

$B_4 = 2\gamma''^3qr^2c + 4pqr^3\gamma''^2$

$B_5 = -4r^3\gamma''^2p^2 - 2\gamma''^3qr^2c + 4r^3\gamma''^2q^2$

$B_6 = -r^2\gamma''^2qa_5 - 2r^3\gamma''\gamma'q^2 - 2r^2\gamma''^2c\gamma q + 2r^3\gamma''\gamma'p^2 + 2r^2\gamma''^2c\gamma'p - 4r^3\gamma''pq\gamma$

Example 1 *From the Kowalevski case, there is a pair $\alpha = ir, \beta = i/2$ which satisfies the system (37) written above. We give two more pairs:*

$$\alpha_1 = 2r(p^2 + q^2) \quad \beta_1 = p^2 + q^2,$$

and

$$\alpha_2 = r\gamma'' \quad \beta_2 = 0.$$

Moreover, any linear combination of the pairs (α, β), (α_1, β_1) and (α_2, β_2) also gives a solution of the system (37) and provides a system with invariant standard measure.

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Elastic deformations

Jurdjevic considered a deformation of the Kowalevski case associated to a Kirchhoff elastic problem, see [23]. The systems are defined by the Hamiltonians

$$H = M_1^2 + M_2^2 + 2M_3^2 + \gamma_1$$

where deformed Poisson structures $\{\cdot,\cdot\}_\tau$ are defined by

$$\{M_i, M_j\}_\tau = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\}_\tau = \epsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\}_\tau = \tau \epsilon_{ijk} M_k,$$

where the deformation parameter takes values $\tau = 0, 1, -1$. The classical Kowalevski case corresponds to the case $\tau = 0$.

Denote

$$e_1 = x_1^2 - (\gamma_1 + i\gamma_2) + \tau$$
$$e_2 = x_2^2 - (\gamma_1 - i\gamma_2) + \tau,$$

where

$$x_{1,2} = \frac{M_1 \pm iM_2}{2}.$$

The integrals of motion

$$I_1 = e_1 e_2$$
$$I_2 = H$$
$$I_3 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3$$
$$I_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \tau(M_1^2 + M_2^2 + M_3^2)$$

may be rewritten in the form (22)

$$k^2 = I_1 = e_1 \cdot e_2$$
$$M_3^2 = e_1 + e_2 + \hat{E}(x_1, x_2)$$
$$M_3\gamma_3 = -x_2 e_1 - x_1 e_2 + \hat{F}(x_1, x_2)$$
$$\gamma_3^2 = x_2^2 e_1 + x_1^2 e_2 + \hat{G}(x_1, x_2),$$

where

$$\hat{G}(x_1, x_2) = -x_1^2 x_2^2 - 2\tau x_1 x_2 - 2\tau(I_1 - \tau) + \tau^2 - I_2$$
$$\hat{F}(x_1, x_2) = (x_1 x_2 + \tau)(x_1 + x_2) + I_3$$
$$\hat{E}(x_1, x_2) = -(x_1 + x_2)^2 + 2(I_1 - \tau).$$

Proposition 9 *Corresponding pencil of conics is determined by equations*

$$a_1 = 0, \; a_5 = 2\tau, \; a_2 = \frac{2(\tau - I_1)}{a_0}, \; a_3 = 2\frac{I_3}{a_0}, \; a_4 = \frac{8\tau(I_1 - \tau) + 4(I_2 - \tau^2)}{a_0}$$

where a_0 is arbitrary.

5 Two-valued groups, Kowalevski equation and Poncelet Porism

5.1 Multivalued groups: defining notions

The structure of multivalued groups was introduced by Buchstaber and Novikov in 1971 (see [5]) in their study of characteristic classes of vector bundles, and it has been studied by Buchstaber and his collaborators since then (see [8] and references therein).

Following [8], we give the definition of an n-valued group on X as a map:

$$m : X \times X \to (X)^n$$
$$m(x, y) = x * y = [z_1, \ldots, z_n],$$

where $(X)^n$ denotes the symmetric n-th power of X and z_i coordinates therein.

Associativity is the condition of equality of two n^2-sets

$$[x * (y * z)_1, \ldots, x * (y * z)_n]$$
$$[(x * y)_1 * z, \ldots, (x * y)_n * z]$$

for all triplets $(x, y, z) \in X^3$.

An element $e \in X$ is *a unit* if

$$e * x = x * e = [x, \ldots, x],$$

for all $x \in X$.

A map inv $: X \to X$ is *an inverse* if it satisfies

$$e \in \mathrm{inv}(x) * x, \quad e \in x * \mathrm{inv}(x),$$

for all $x \in X$.

Following Buchstaber, we say that m defines *an n-valued group structure* (X, m, e, inv) if it is associative, with a unit and an inverse.

An n-valued group X acts on the set Y if there is a mapping

$$\phi : X \times Y \to (Y)^n$$
$$\phi(x, y) = x \circ y,$$

such that the two n^2-multisubsets of Y

$$x_1 \circ (x_2 \circ y) \quad (x_1 * x_2) \circ y$$

are equal for all $x_1, x_2 \in X, y \in Y$. It is additionally required that

$$e \circ y = [y, \ldots, y]$$

for all $y \in Y$.

Example 2 (A two-valued group structure on $\mathbb{Z}_+$, [7]) *Let us consider the set of nonnegative integers $\mathbb{Z}_+$ and define a mapping*

$$m : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2,$$
$$m(x,y) = [x+y, |x-y|].$$

This mapping provides a structure of a two-valued group on $\mathbb{Z}_+$ with the unit $e = 0$ and the inverse equal to the identity $\mathrm{inv}(x) = x$.

In [7] sequence of two-valued mappings associated with the Poncelet porism was identified as the algebraic representation of this 2-valued group. Moreover, the algebraic action of this group on $\mathbb{CP}^1$ was studied and it was shown that in the irreducible case all such actions are generated by Euler-Chasles correspondences.

In the sequel, we are going to show that there is another 2-valued group and its action on $\mathbb{CP}^1$ which is even more closely related to the Euler-Chasles correspondence and to the Great Poncelet Theorem, and which is at the same time intimately related to the Kowalevski fundamental equation and to the Kowalevski change of variables.

However, we will start our approach with a simple example.

5.2   The simplest case: 2-valued group p_2

Among the basic examples of multivalued groups, there are n-valued additive group structures on $\mathbb{C}$. For $n = 2$, this is a two-valued group p_2 defined by the relation

$$m_2 : \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^2$$
$$x *_2 y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2] \tag{39}$$

The product $x *_2 y$ corresponds to the roots in z of the polynomial equation

$$p_2(z, x, y) = 0,$$

where

$$p_2(z, x, y) = (x + y + z)^2 - 4(xy + yz + zx).$$

Our starting point in this section is the following

Lemma 7 *The polynomial $p_2(z, x, y)$ is discriminantly separable. The discriminants satisfy relations*

$$\mathcal{D}_z(p_2)(x, y) = P(x)P(y) \quad \mathcal{D}_x(p_2)(y, z) = P(y)P(z) \quad D_y(p_2)(x, z) = P(x)P(z),$$

where

$$P(x) = 2x.$$

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The polynomial p_2 as discriminantly separable, generates a case of generalized Kowalevski system of differential equations, but this time with $K = 0$. The system is defined by

$$\hat{E} = 0 \quad \hat{F} = 1 \quad \hat{G} = 0, \tag{40}$$

and the equations of motion have the form

$$
\begin{aligned}
\frac{de_1}{dt} &= -\alpha e_1 \\
\frac{de_2}{dt} &= \alpha e_2 \\
\frac{dx_1}{dt} &= -\beta(rx_1 + cg) \\
\frac{dx_2}{dt} &= \beta(rx_2 + cg) \\
\frac{dr}{dt} &= -\frac{\alpha}{2r}(e_1 - e_2) \\
\frac{dg}{dt} &= 2\beta c + \frac{(2r\beta - \alpha)}{2c^2 g}\left(e_1 x_2^2 - e_2 x_1^2\right)
\end{aligned}
\tag{41}
$$

In the standard rigid-body coordinates with $\alpha = ir$, $\beta = i/2$ the last two equations become

$$\dot{r} = 2pq + c\gamma' \quad \dot{\gamma}'' = ic.$$

Lemma 8 *The integrals of the system defined by the equations (40) are*

$$
\begin{aligned}
k^2 &= e_1 e_2 \\
r^2 &= e_1 + e_2 \\
crg &= 1 - x_1 e_2 - x_2 e_1 \\
c^2 g^2 &= x_2^2 e_1 + x_1^2 e_2
\end{aligned}
$$

From the last Lemma 8 we get the relation

$$2e_1 x_2 + 2e_2 x_1 - 1 + k^2(x_1 - x_2)^2 = 0.$$

Now, together with the first integral relation from the Lemma 8, similar as in the Kowalevski case, we get

$$\left[\sqrt{e_1}\frac{\sqrt{2x_2}}{x_1 - x_2} \pm \sqrt{e_2}\frac{\sqrt{2x_1}}{x_1 - x_2}\right]^2 = (w_1 \pm k)(w_2 \mp k), \tag{42}$$

where w_1, w_2 are solutions of the quadratic equation

$$F_2(w, x_1, x_2) := (x_1 - x_2)^2 w^2 - 2(x_1 + x_2)w + 1 = 0. \tag{43}$$

The polynomial F_2 is obtained by transposition from the polynomial p_2 and, thus, it is discriminantly separable:

$$\mathcal{D}_x(F_2)(y, z) = P(y)\varphi(z),$$

where

$$\varphi(z) = z^3.$$

Following lines of integration, we finally come to

Proposition 10 *The system of differential equations defined by 40 is integrated to through the solutions of the system*

$$
\begin{aligned}
\frac{ds_1}{s_1\sqrt{\Phi_1(s_1)}} + \frac{ds_2}{s_2\sqrt{\Phi_1(s_2)}} &= 0 \\
\frac{ds_1}{\sqrt{\Phi_1(s_1)}} + \frac{ds_2}{\sqrt{\Phi_1(s_2)}} &= \frac{i}{2}dt,
\end{aligned}
\tag{44}
$$

where

$$\Phi(s) = s(s - e_4)(s - e_5)$$

is the polynomial of degree 3.

Similar systems appeared in a slightly different context in the works of Appel'rot, Mlodzeevskii, Delone in their study of degenerations of the Kowalevski top (see [1], [29],[11]). In particular, we may construct *Delone-type* solutions of the last system:

$$s_1 = 0, \quad s_2 = \wp\left(\frac{i}{4}(t - t_0)\right).$$

We can also consider integrable perturbation of the previous integrable system, defined by:

$$
\begin{aligned}
\hat{E} &= k_1 - 2a_1(x_1 + x_2) \\
\hat{F} &= k_2 + \frac{a_5}{2}(x_1 + x_2) + a_1 x_1 x_2 \\
\hat{G} &= k_3 - a_5 x_1 x_2.
\end{aligned}
\tag{45}
$$

The equations of motion have the form

$$\frac{de_1}{dt} = -\alpha e_1$$

$$\frac{de_2}{dt} = \alpha e_2$$

$$\frac{dx_1}{dt} = -\beta(rx_1 + cg)$$

$$\frac{dx_2}{dt} = \beta(rx_2 + cg)$$

$$\frac{dr}{dt} = -\frac{\alpha}{2r}(e_1 - e_2) - \frac{a_1}{2}\beta(x_2 - x_1)$$

$$\frac{dg}{dt} = 2\beta c + \frac{(2r\beta - \alpha)}{2c^2 g}\left(e_1 x_2^2 - e_2 x_1^2\right) + \frac{a_5}{2}c\beta(x_2 - x_1)$$

$$(46)$$

In the standard rigid-body coordinates with $\alpha = ir$, $\beta = i/2$ the last two equations become

$$\dot{r} = 2pq + c\gamma' + \frac{a_1}{2}q$$

$$\dot{\gamma}'' = ic(1 + i\frac{a_5}{2}q).$$

Corresponding polynomial

$$F(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2)$$

where

$$R(x_1, x_2) = \hat{E}x_1 x_2 + \hat{F}(x_1 + x_2) + \hat{G}, \quad R_1(x_1, x_2) = \hat{E}\hat{F} - \hat{G}^2,$$

is discriminantly separable and

$$\mathcal{D}_{x_1}(s, x_2) = \varphi(s)P(x_2),$$

where

$$\varphi(s) = (2s - a_5)(2a_1 + a_5 s - 2s^2)$$

$$P(x) = 2x(2a_1 x^2 - a_5 x - 2).$$

5.3 2-valued group structure on $\mathbb{CP}^1$, the Kowalevski fundamental equation and Poncelet porism

Now we pass to the general case. We are going to show that the general pencil equation represents an action of a two valued group structure. Recognition of this structure enables us to give to 'the mysterious Kowalevski change of variables' a final algebro-geometric expression and explanation, developing further the ideas of Weil and Jurdjevic (see [33], [23]). Amazingly, the associativity condition for this action from geometric point of view is nothing else than the Great Poncelet Theorem for a triangle.

As we have already mentioned, the general pencil equation

$$F(s, x_1, x_2) = 0$$

is connected with two isomorphic elliptic curves

$$\Gamma_1 : y^2 = P(x)$$
$$\Gamma_2 : t^2 = J(s)$$

where the polynomials P, J of degree four and three respectively are defined by the equations (13). Suppose that the cubic one Γ_2 is rewritten in the canonical form

$$\Gamma_2 : t^2 = J'(s) = 4s^3 - g_2 s - g_3.$$

Moreover, denote by $\psi : \Gamma_2 \to \Gamma_1$ a birational morphism between the curves induced by a fractional-linear transformation $\hat{\psi}$ which maps three zeros of J' and ∞ to the four zeros of the polynomial P.

The curve Γ_2 as a cubic curve has the group structure. Together with its subgroup $\mathbb{Z}_2$ it defines the standard two-valued group structure of coset type on $\mathbb{CP}^1$ (see [6], [8]):

$$s_1 *_c s_2 = \left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right], \quad (47)$$

where $t_i = J'(s_i), i = 1, 2$.

Theorem 4 *The general pencil equation after fractional-linear transformations*

$$F(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

defines the two valued coset group structure $(\Gamma_2, \mathbb{Z}_2)$ defined by the relation (47).

Proof. After the fractional-linear transformations, the pencil equation obtains the form

$$F_1(s, x, y) = T(s, x)y^2 + V(s, x)y + W(s, x),$$

where

$$T(s, x) = -4s^2 + 4sx - s^2$$
$$V(s, x) = 4sx^2 + 2s^2 x - 2xg_2 - g_2 s - 4g_3$$
$$W(s, x) = -s^2 x^2 - g_2 xs - 4xg_3 - 2g_3 s - \frac{g_2^2}{4}.$$

We apply now a linear change of variables γ on s:

$$m = \gamma(s) := \frac{s}{2}$$

and get

$$F_2(m, x, y) = F_1(2m, x, y).$$

Denote by $P = (m, n)$ and $M = (x, u)$ two arbitrary points on the curve Γ_2, which means

$$n^2 = 4m^3 - g_2 m - g_3$$
$$u^2 = 4x^3 - g_2 x - g_3.$$

We want to find points $N_1 = (y_1, v_1)$ and $N_2 = (y_2, v_2)$ on Γ_2 which correspond by F_2 to P and M. These points are

$$y_1 = \frac{-V(s, x) + 4nu}{2T(s, x)} \quad v_1 = -\frac{2xT(s, y_1) + V(s, y_1)}{4n}$$
$$y_2 = \frac{-V(s, x) - 4nu}{2T(s, x)} \quad v_2 = -\frac{2xT(s, y_2) + V(s, y_2)}{4n}$$

By trivial algebraic transformations

$$y_1 = \frac{-4mx^2 - 4xm^2 + xg_2 + mg_2 + 2g_3 + 2nu}{-4(x - m)^2}$$
$$= \frac{-4mx(x + m) + x^3 + m^3 - x^3 + xg_2 + g_3 - m^3 + mg_2 + g_3 + 2nu}{-4(x - m)^2}$$
$$= -x - m + \left(\frac{u - n}{2(x - m)}\right)^2$$

we get the first part of the operation of the two-valued group $(\Gamma_2, \mathbb{Z}_2)$ defined by the relation (47). Applying similar transformations to y_2 we get the second part of the relation (47) as well. This ends the proof of the Theorem. $\qquad\square$

The Kowalevski change of variables (see equations (31)) is infinitesimal of the correspondence which maps a pair of points (M_1, M_2) from the curve Γ_1 to a pair of points (S_1, S_2) of the curve Γ_2. One view to this correspondence has been given in [23] following Weil [33]. In our approach, there is a geometric view to this mapping as the correspondence which maps *two tangents to the conic C to the pair of conics from the pencil which contain the intersection point of the two lines.*

If we apply fractional-linear transformations to transform the curve Γ_1 into the curve Γ_2, then the above correspondence is nothing else then the two-valued group operation $*_c$ on $(\Gamma_2, \mathbb{Z}_2)$.

Theorem 5 *The Kowalevski change of variables is equivalent to infinitesimal of the action of the two valued coset group $(\Gamma_2, \mathbb{Z}_2)$ on Γ_1. Up to the fractional-linear transformation, it is equivalent to the operation of the two valued group $(\Gamma_2, \mathbb{Z}_2)$.*

Now, the Kotter trick from the Section 4 (see the equations (28, 29) can be presented as a commutative diagram.

Proposition 11 *The Kotter transformation defined by the equations (28, 29) makes the following diagram commutative:*

$$
\begin{array}{ccccc}
\mathbb{C}^4 & \xrightarrow{\;i_{\Gamma_1}\times i_{\Gamma_1}\times m\;} & \Gamma_1\times\Gamma_1\times\mathbb{C} & \xrightarrow{\;\psi^{-1}\times\psi^{-1}\times id\;} & \Gamma_2\times\Gamma_2\times\mathbb{C}
\end{array}
$$

The mappings are arranged as the commutative diagram:

- $i_{\Gamma_1}\times i_{\Gamma_1}\times id\times id$: $\mathbb{C}^4 \to \Gamma_1\times\Gamma_1\times\mathbb{C}\times\mathbb{C}$
- $i_a\times i_a\times m$: $\Gamma_1\times\Gamma_1\times\mathbb{C} \to \mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{C}$
- $p_1\times p_1\times id$
- $\varphi_1\times\varphi_2$: $\Gamma_1\times\Gamma_1\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C}$
- $\hat\psi^{-1}\times\hat\psi^{-1}\times id$: $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{C} \to \mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{C}$
- $p_1\times p_1\times id$
- m_2 : $\mathbb{C}\times\mathbb{C}\to\mathbb{CP}^2$
- $m_c\times\tau_c$: $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{C}\to\mathbb{CP}^2\times\mathbb{C}/\sim$
- f : $\mathbb{CP}^2\times\mathbb{C}/\sim \;\to\; \mathbb{CP}^2$

The mappings are defined as follows

$$i_{\Gamma_1} : x \mapsto (x, \sqrt{P(x)})$$
$$m : (x,y) \mapsto x\cdot y$$
$$i_a : x \mapsto (x, 1)$$
$$p_1 : (x,y) \mapsto x$$
$$m_c : (x,y) \mapsto x *_c y$$
$$\tau_c : x \mapsto (\sqrt{x}, -\sqrt{x})$$
$$\varphi_1 : (x_1, x_2, e_1, e_2) \mapsto \sqrt{e_1}\,\frac{\sqrt{P(x_2)}}{x_1 - x_2}$$
$$\varphi_2 : (x_1, x_2, e_1, e_2) \mapsto \sqrt{e_2}\,\frac{\sqrt{P(x_1)}}{x_1 - x_2}$$
$$f : ((s_1, s_2, 1), (k, -k)) \mapsto [(\gamma^{-1}(s_1) + k)(\gamma^{-1}(s_2) - k), (\gamma^{-1}(s_2) + k)(\gamma^{-1}(s_1) - k)]$$

From the Proposition 11 we see that the two-valued group plays an important role in the Kowalevski system and its generalizations.

Putting together the geometric meaning of the pencil equation and algebraic structure of the two valued group we come to the connection with the Great Poncelet Theorem ([30], see also [3], [15] and [16]). For the reader's sake we are

going to formulate the Great Poncelet Theorem for triangles in the form we are going to use below.

Theorem 6 (Great Poncelet Theorem for triangles [30]) *Given four conics C_1, C_2, C_3, C from a pencil and three lines a_1, a_2, a_3, tangents to the conic C such that a_1, a_2 intersect on C_1, a_2, a_3 intersect on C_2 and a_2, a_3 intersect on C_3. Moreover, we suppose that the tangents to the conics C_1, C_2, C_3 at the intersection points are not concurrent. Given b_1, b_2 tangents to the conic C which intersect at C_1. Then there exists b_3, tangent to the conic C such that the triplet (b_1, b_2, b_3) satisfies all conditions as (a_1, a_2, a_3).*

Now, we are going back to the associativity condition for the action of the double-valued group $(\Gamma_2, \mathbb{Z}_2)$.

Theorem 7 *Associativity conditions for the group structure of the two-valued coset group $(\Gamma_2, \mathbb{Z}_2)$ and for its action on Γ_1 are equivalent to the great Poncelet theorem for a triangle.*

Proof. Denote by P and Q two arbitrary elements of the two-valued group $(\Gamma_2, \mathbb{Z}_2)$ and M an arbitrary point on the curve Γ_1. Let

$$Q * P = [P_1, P_2]$$

and

$$P \circ M = [N_1, N_2].$$

Associativity means the equality of the two quadruples:

$$[Q \circ N_1, Q \circ N_2] = [P_1 \circ M, P_2 \circ M].$$

Let us consider previous situation from geometric point of view. Recall the geometric meaning of the equation of a pencil of conics

$$F(s, x_1, x_2) = 0.$$

Variables x_1 and x_2 denote the Darboux coordinates of two tangents to the conic C_2 which intersect at the conic C_s with the pencil parameter equal to s.

Denote by C_P and C_Q the conics from the pencil which correspond to the elements P, Q, and by l_M, l_{N_1}, l_{N_2} the tangents to the conic C_2 which correspond to the points M, N_1, N_2 of the curve Γ_1. Then, l_{N_1} and l_{N_2} are the two lines tangent to C_2 which intersect l_M at the conic C_P.

Moreover, if we denote

$$Q \circ N_1 = [N_3, N_4], \quad Q \circ N_2 = [N_5, N_6]$$

then corresponding lines $l_{N_3}, l_{N_4}, l_{N_5}, l_{N_6}$, tangent to the conic C_2 satisfy the conditions: the pairs of lines (l_{N_1}, l_{N_3}), (l_{N_1}, l_{N_4}), (l_{N_2}, l_{N_5}), (l_{N_2}, l_{N_6}) all intersect at the conic C_Q.
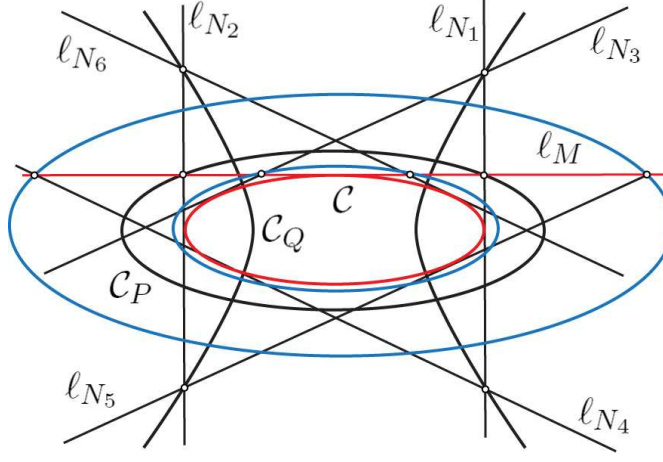
Figure 1: Associativity condition and Poncelet theorem

Now, associativity of the action is equivalent to the existence of *a pair of conics* (C_{P_1}, C_{P_2}) such that (l_M, l_{N_3}) and (l_M, l_{N_6}) intersect at the conic C_{P_1}, while (l_M, l_{N_5}) and (l_M, l_{N_4}) intersect at the conic C_{P_2}, see the Fig. 5.3.

Consider the intersection of the lines (l_M, l_{N_3}). Choose the conic from the pencil which contains the intersection point, such that the tangent to this conic at the intersection point is not concurrent with the tangents to the conics C_P and C_Q at the intersection points (l_M, l_{N_1}) and (l_{N_1}, l_{N_3}) respectively. Denote the conic C_{P_1}. Then by applying Great Poncelet Theorem for triangle (see the Theorem above, [30],see also [3], [15], [16]), one of the lines l_{N_5} and l_{N_6}, say the last one, intersects L_M at the conic C_{P_1}. The tangent to this conic at the intersection point is not concurrent with the tangents to the conics C_P and C_Q at the intersection points (l_M, l_{N_2}) and (l_{N_2}, l_{N_6}) respectively.

In the same way, by considering intersection of the lines (l_M, l_{N_4}) we come to the conic (C_{P_2}) from the pencil, which, by Great Poncelet Theorem contains intersections of (l_M, l_{N_4}) and (l_M, l_{N_5}).

Since the result of the operation in the double-valued group between elements P, Q doesn't depend on the choice of the point M to which the action is applied, the conics C_{P_2} and C_{P_1} in the previous construction should not depend of the choice of the line l_M. This independence is equivalent to **the poristic** nature of the Poncelet Theorem. This demonstrates the equivalence between the associativity condition and the Great Poncelet Theorem for a triangle. $\square$

From the last two theorems we get finally

Conclusion *Geometric settings for the Kowalevski change of variables is the Great Poncelet Theorem for a triangle.*

Acknowledgements

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